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EACH COMPLETE BIPARTITE GRAPH MINUS A MATCHING IS REPRESENTABLE BY LINE SEGMENTS

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Each complete bipartite graph minus a matching is representable by line segments*)

by

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ABSTRACT

It is proved that each complete bipartite graph minus a matching is the intersection graph of a collection of line segments.

KEY WORDS & PHRASES: *Intersection graph (representative graph), convex sets, line segments.*

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*) This report is not for review; it is meant for publication elsewhere.

In [1] M. LAS VERGNAS and L. LOVÁSZ posed the following problem: Is $K_{7,7}$ minus a perfect matching representable by convex sets in the plane $\mathbb{R}^2$? Here, as usual, a graph G is called *representable* by sets of a special kind if G is the representative graph (or intersection graph) of a collection of sets of that kind.

Let G_n be the graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching. LAS VERGNAS observed that each G_n is representable by arcwise connected subsets of $\mathbb{R}^2$, and that G_6 is representable by convex subsets of $\mathbb{R}^2$. Subsequently, the third author showed that G_7 is representable by semilines. J BECK (private communication via LOVÁSZ) proved independently that even G_8 is representable by semilines, and this inspired LAS VERGNAS (private communication) to arrange a construction of line segments with intersection graph G_{10}.

In this note we prove that each G_n is representable by semilines (or by line segments, or by convex subsets) in $\mathbb{R}^2$. In fact, we show that $G_{\aleph_0}$ is representable by line segments or by semilines. A consequence of this is that each graph obtained from some complete bipartite graph by removing a (not necessarily perfect) matching is representable by line segments, since each such graph is an induced subgraph of a G_n.

Our construction is as follows. Let C be the curve $\{(x,\tan x) \mid -\frac{1}{2}\pi < x < +\frac{1}{2}\pi\}$. Define, for each natural number n, points P_n and P'_n on the curve C, and line segments L_n and L'_n inductively by:

(i) $P_1 = (\frac{1}{4}\pi, \tan \frac{1}{4}\pi)$, $P'_1 = (-\frac{1}{4}\pi, -\tan \frac{1}{4}\pi)$;

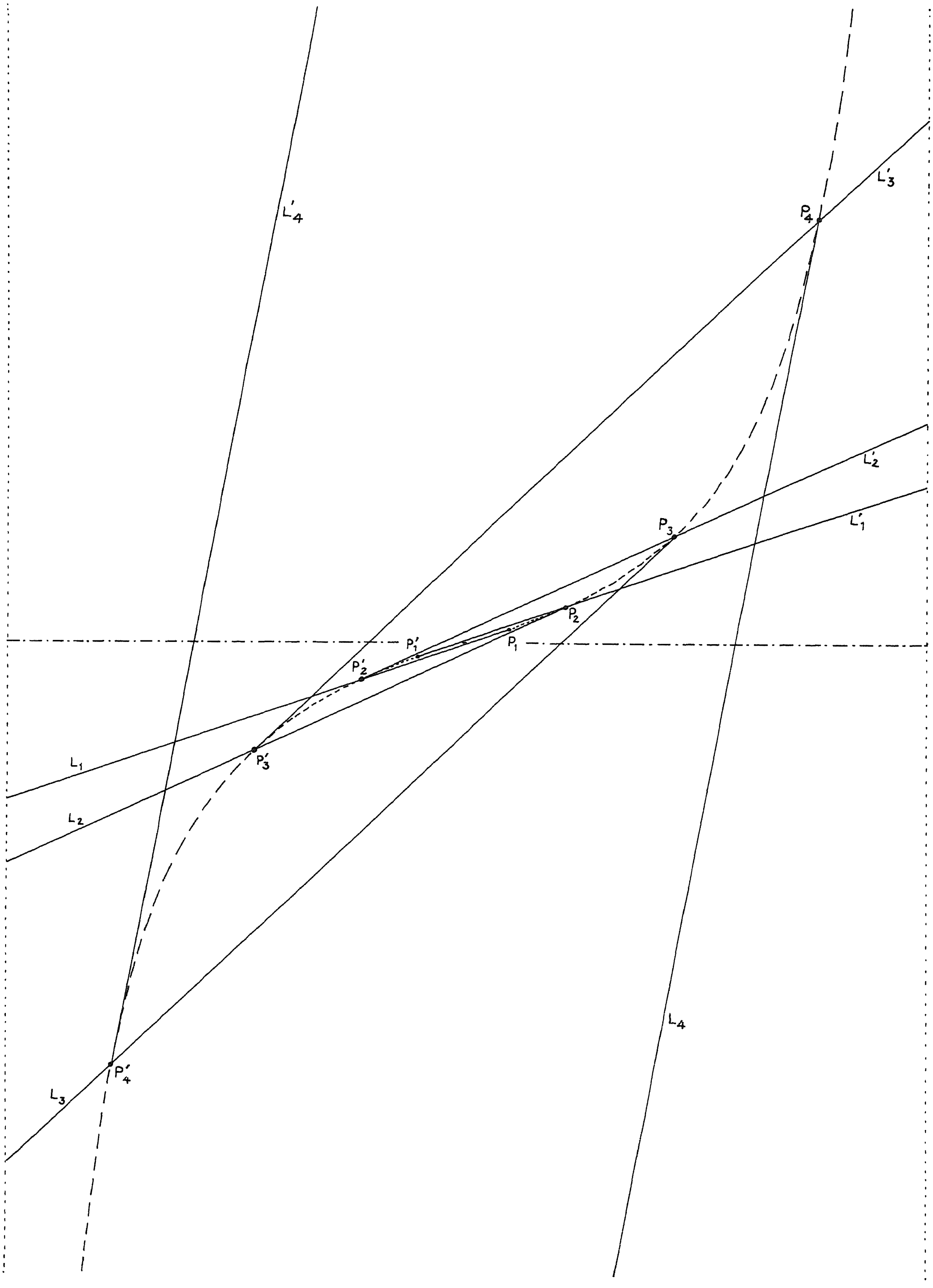
(ii) if $P_1,\ldots,P_n$, $P'_1,\ldots,P'_n$ and $L_1,\ldots,L_{n-1}$, $L'_1,\ldots,L'_{n-1}$ ($n \geq 1$) are defined, then define line segments L_n and L'_n, and points P_{n+1} and P'_{n+1} as follows:

L_n is the segment of the tangent of C in P_n with end points P_n and the intersection point with the vertical line "$x = -\frac{1}{2}\pi$";

L'_n is the segment of the tangent of C in P'_n with end points P'_n and the intersection point with the vertical line "$x = +\frac{1}{2}\pi$";

P_{n+1} is the intersection point of L'_n and the curve C;

P'_{n+1} is the intersection point of L_n and the curve C.



L'4
P4
L'3
L'2
P3
L'1
P2
P'1
P1
P'2
L1
P'3
L2
L4
P'4
L3

It can easily be seen that the line segments $L_1, L_2, \ldots$ and $L'_1, L'_2, \ldots$ satisfy the following conditions: if $n, m \in \mathbb{N}$, $n \neq m$, then $L_n \cap L_m = L'_n \cap L'_m = \emptyset \neq L_n \cap L'_m$, and if $n \in \mathbb{N}$, then $L_n \cap L'_n = \emptyset$. Hence this collection of line segments has as intersection graph the graph $G_{\aleph_0}$. Also it is clear that by extending the line segments L_n beyond the line "$x = -\frac{1}{2}\pi$" and the line segments L'_n beyond the line "$x = +\frac{1}{2}\pi$" we obtain a collection of semilines, again with intersection graph $G_{\aleph_0}$.

The problem of representing G_7 by convex sets in $\mathbb{R}^2$ arose from the problem of characterizing all minimal graphs not representable by convex sets in $\mathbb{R}^2$ (similar to the BOLAND & LEKKERKERKER [2] characterization of minimal graphs not representable by convex sets (intervals) in $\mathbb{R}$). This general question is still unanswered.

REFERENCES

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[2] BOLAND J. & C. LEKKERKERKER, *Representation of a finite graph by a set of intervals on the real line,* Fund. Math. 51 (1962) 45-64.